AP Physics C: Prep
Homework for Day 6

Attached to this document are scans of 4 pages from a calculus textbook (Calculus, 5th edition, late trigonometry version by Swokowski (1992)). Please do the following problems on a separate sheet of paper. On these pages, be sure to include not only the problem number, but also the section that it comes from. All of these problems are applications of the chain rule.

Homework for Session 1:

- Exercises 3.5, problems 14, 20, 22, and 24
- Exercises 7.2, problems 10, 12, 14, and 23
- Exercises 7.3, problems 6, 10, 13, and 14
- Exercises 8.2, problems 18, 19, 20, and 25

Homework for Session 2:

- Exercises 3.5, problems 12, 19, 21, and 26
- Exercises 7.2, problems 9, 13, 15, and 24
- Exercises 7.3, problems 5, 7, 15, and 16
- Exercises 8.2, problems 17, 19, 20, and 25

These problems are circled and the corresponding session is written next to the problem number.

Calculus, 5th Ed. (1992) by Swokowski

EXERCISES 3.5

Exer. 1–4: Use the chain rule to find $\frac{dy}{dx}$, and express the answer in terms of x.

1 $y = u^2$; $\quad u = x^3 - 4$

2 $y = \sqrt[3]{u}$; $\quad u = x^2 + 5x$

3 $y = 1/u$; $\quad u = \sqrt{3x - 2}$

4 $y = 3u^2 + 2u$; $\quad u = 4x$

Exer. 5–26: Find the derivative.

5 $f(x) = (x^2 - 3x + 8)^3$

6 $f(x) = (4x^3 + 2x^2 - x - 3)^2$

7 $g(x) = (8x - 7)^{-5}$

8 $k(x) = (5x^2 - 2x + 1)^{-3}$

9 $f(x) = \dfrac{x}{(x^2 - 1)^4}$

10 $g(x) = \dfrac{x^4 - 3x^2 + 1}{(2x + 3)^4}$

11 $f(x) = (8x^3 - 2x^2 + x - 7)^5$

2 12 $g(w) = (w^4 - 8w^2 + 15)^4$

13 $F(v) = (17v - 5)^{1000}$

1 14 $s(t) = (4t^5 - 3t^3 + 2t)^{-2}$

15 $N(x) = (6x - 7)^3(8x^2 + 9)^2$

16 $f(w) = (2w^2 - 3w + 1)(3w + 2)^4$

17 $g(z) = \left(z^2 - \dfrac{1}{z^2}\right)^6$

18 $S(t) = \left(\dfrac{3t + 4}{6t - 7}\right)^3$

2 19 $k(r) = \sqrt[3]{8r^3 + 27}$

1 20 $h(z) = (2z^2 - 9z + 8)^{-2/3}$

2 21 $F(v) = \dfrac{5}{\sqrt[5]{v^5 - 32}}$

1 22 $k(s) = \dfrac{1}{\sqrt{3s - 4}}$

23 $g(w) = \dfrac{w^2 - 4w + 3}{w^{3/2}}$

1 24 $K(x) = \sqrt{4x^2 + 2x + 3}$

25 $H(x) = \dfrac{2x + 3}{\sqrt{4x^2 + 9}}$

2 26 $f(x) = (7x + \sqrt{x^2 + 3})^6$

Exer. 27–30: (a) Find equations of the tangent line and the normal line to the graph of the equation at P. (b) Find the x-coordinates on the graph at which the tangent line is horizontal.

27 $y = (4x^2 - 8x + 3)^4$; $\quad P(2, 81)$

28 $y = (2x - 1)^{10}$; $\quad P(1, 1)$

29 $y = \left(x + \dfrac{1}{x}\right)^5$; $\quad P(1, 32)$

30 $y = \sqrt{2x^2 + 1}$; $\quad P(-1, \sqrt{3})$

Exer. 31–34: Find the first and second derivatives.

31 $g(z) = \sqrt{3z + 1}$

32 $k(s) = (s^2 + 4)^{2/3}$

33 $k(r) = (4r + 7)^5$

34 $f(x) = \sqrt[5]{10x + 7}$

Exer. 35–36: Use differentials to approximate the value.

35 $\sqrt[3]{65}$ (*Hint:* Let $y = \sqrt[3]{x}$.)

36 $\sqrt[5]{35}$

37 If an object of mass m has velocity v, then its *kinetic energy* K is given by $K = \frac{1}{2}mv^2$. If v is a function of time t, use the chain rule to find a formula for dK/dt.

38 As a spherical weather balloon is being inflated, its radius r is a function of time t. If V is the volume of the balloon, use the chain rule to find a formula for dV/dt.

39 When a space shuttle is launched into space, an astronaut's body weight decreases until a state of weightlessness is achieved. The weight W of a 150-pound astronaut at an altitude of x kilometers above sea level is given by

$$W = 150\left(\frac{6400}{6400 + x}\right)^2.$$

If the space shuttle is moving away from the earth's surface at the rate of 6 km/sec, at what rate is W decreasing when $x = 1000$ km?

40 The length-weight relationship for Pacific halibut is well described by the formula $W = 10.375L^3$, where L is the length in meters and W is the weight in kilograms. The rate of growth in length dL/dt is given by $0.18(2 - L)$, where t is time in years.

(a) Find a formula for the rate of growth in weight dW/dt in terms of L.

(b) Use the formula in part (a) to estimate the rate of growth in weight of a halibut weighing 20 kilograms.

41 If $k(x) = f(g(x))$ and if $f(2) = -4$, $g(2) = 2$, $f'(2) = 3$, and $g'(2) = 5$, find $k(2)$ and $k'(2)$.

42 Let p, q, and r be functions such that $p(z) = q(r(z))$. If $r(3) = 3$, $q(3) = -2$, $r'(3) = 4$, and $q'(3) = 6$, find $p(3)$ and $p'(3)$.

43 If $f(t) = g(h(t))$ and if $f(4) = 3$, $g(4) = 3$, $h(4) = 4$, $f'(4) = 2$, and $g'(4) = -5$, find $h'(4)$.

44 If $u(x) = v(w(x))$ and if $v(0) = -1$, $w(0) = 0$, $u(0) = -1$, $v'(0) = -3$, and $u'(0) = 2$, find $w'(0)$.

FIGURE 7.10

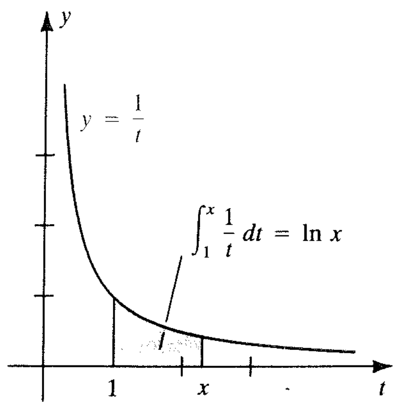
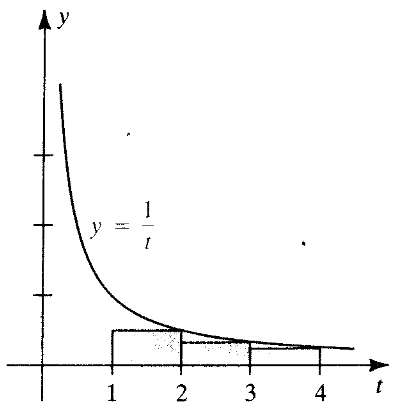


FIGURE 7.11



area of the region shown in Figure 7.10. The sum of the areas of the three rectangles shown in Figure 7.11 is

$$\tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} = \tfrac{13}{12}.$$

Since the area under the graph of $y = 1/t$ from $t = 1$ to $t = 4$ is $\ln 4$, we see that

$$\ln 4 > \tfrac{13}{12} > 1.$$

It follows that if M is any positive rational number, then

$$M \ln 4 > M, \quad \text{or} \quad \ln 4^M > M.$$

If $x > 4^M$, then since ln is an increasing function,

$$\ln x > \ln 4^M > M.$$

This proves that $\ln x$ can be made as large as desired by choosing x sufficiently large; that is,

$$\lim_{x \to \infty} \ln x = \infty.$$

To investigate the case $x \to 0^+$, we first note that

$$\ln \frac{1}{x} = \ln 1 - \ln x = 0 - \ln x = -\ln x.$$

Hence $$\lim_{x \to 0^+} \ln x = \lim_{x \to 0^+} \left(-\ln \frac{1}{x}\right).$$

As x approaches zero through positive values, $1/x$ increases without bound and, therefore, so does $\ln (1/x)$. Consequently, $-\ln (1/x)$ *decreases* without bound; that is,

$$\lim_{x \to 0^+} \ln x = -\infty.$$

EXERCISES 7.2

Exer. 1–24: Find $f'(x)$ if $f(x)$ is the given expression.

1 $\ln (9x + 4)$

2 $\ln (x^4 + 1)$

3 $\ln (3x^2 - 2x + 1)$

4 $\ln (4x^3 - x^2 + 2)$

5 $\ln |3 - 2x|$

6 $\ln |4 - 3x|$

7 $\ln |2 - 3x|^5$

8 $\ln |5x^2 - 1|^3$

2 9 $\ln \sqrt{7 - 2x^3}$

1 10 $\ln \sqrt[3]{6x + 7}$

11 $x \ln x$

1 12 $\ln (\ln x)$

2 13 $\ln \sqrt{x} + \sqrt{\ln x}$

1 14 $\ln x^3 + (\ln x)^3$

2 15 $\dfrac{1}{\ln x} + \ln \dfrac{1}{x}$

16 $\dfrac{x^2}{\ln x}$

17 $\ln [(5x - 7)^4 (2x + 3)^3]$

18 $\ln [\sqrt[3]{4x - 5}\,(3x + 8)^2]$

19 $\ln \dfrac{\sqrt{x^2 + 1}}{(9x - 4)^2}$

20 $\ln \dfrac{x^2 (2x - 1)^3}{(x + 5)^2}$

21 $\ln \sqrt{\dfrac{x^2 - 1}{x^2 + 1}}$

22 $\ln \sqrt{\dfrac{4 + x^2}{4 - x^2}}$

1 23 $\ln (x + \sqrt{x^2 - 1})$

2 24 $\ln (x + \sqrt{x^2 + 1})$

Exer. 25–28: Use implicit differentiation to find y'.

25 $3y - x^2 + \ln xy = 2$

26 $y^2 + \ln (x/y) - 4x = -3$

27 $x \ln y - y \ln x = 1$

28 $y^3 + x^2 \ln y = 5x + 3$

Exer. 29–34: Use logarithmic differentiation to find dy/dx.

29 $y = (5x + 2)^3 (6x + 1)^2$

EXAMPLE 4 If each cell of a tumor has two targets, then the two target–one hit surviving fraction is given by

$$f(x) = 1 - (1 - e^{-kx})^2,$$

where k is the average size of a cell. Analyze the graph of f to determine what effect increasing the dosage x has on decreasing the surviving fraction of tumor cells.

SOLUTION First note that if $x = 0$, then $f(0) = 1$; that is, if there is no dose, then all cells survive. Differentiating, we obtain

$$\begin{aligned} f'(x) &= 0 - 2(1 - e^{-kx})\, D_x\,(1 - e^{-kx}) \\ &= -2(1 - e^{-kx})(ke^{-kx}) \\ &= -2ke^{-kx}(1 - e^{-kx}). \end{aligned}$$

Since $f'(x) < 0$ for every $x > 0$ and $f'(0) = 0$, the function f is decreasing and the graph has a horizontal tangent line at the point $(0, 1)$. We may verify that the second derivative is

$$f''(x) = 2k^2e^{-kx}(1 - 2e^{-kx}).$$

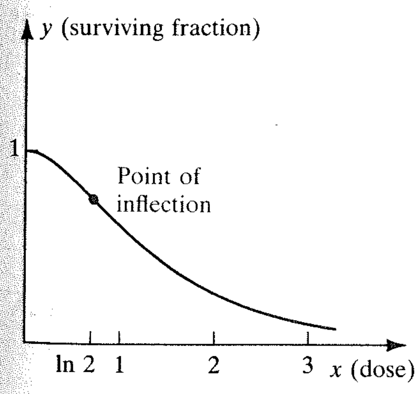
We see that $f''(x) = 0$ if $1 - 2e^{-kx} = 0$ (that is, if $e^{-kx} = \frac{1}{2}$, or, equivalently, $-kx = \ln \frac{1}{2} = -\ln 2$). This gives us

$$x = \frac{1}{k} \ln 2.$$

It can be verified that if $0 \le x < (1/k) \ln 2$, then $f''(x) < 0$, and hence the graph is concave downward. If $x > (1/k) \ln 2$, then $f''(x) > 0$, and the graph is concave upward. This implies that there is a point of inflection with x-coordinate $(1/k) \ln 2$. The y-coordinate of this point is

$$\begin{aligned} f\left(\frac{1}{k} \ln 2\right) &= 1 - (1 - e^{-\ln 2})^2 \\ &= 1 - (1 - \tfrac{1}{2})^2 = \tfrac{3}{4}. \end{aligned}$$

FIGURE 7.14
Surviving fraction of tumor cells after a radiation treatment



The graph is sketched in Figure 7.14 for the case $k = 1$. The *shoulder* on the curve near the point $(0, 1)$ represents the threshold nature of the treatment; that is, a small dose results in very little tumor elimination. Note that if x is large, then an increase in dosage has little effect on the surviving fraction. To determine the ideal dose that should be administered to a given patient, specialists in radiation therapy must also take into account the number of healthy cells that are killed during a treatment.

EXERCISES 7.3

Exer. 1–20: Find $f'(x)$ if $f(x)$ equals the given expression.

1 e^{-5x}	**2** e^{3x}	**11** $e^x/(x^2+1)$	**12** $x/e^{(x^2)}$
3 e^{3x^2}	**4** e^{1-x^3}	1 **13** $(e^{4x}-5)^3$	1 **14** $(e^{3x}-e^{-3x})^4$
2 **5** $\sqrt{1+e^{2x}}$	1 **6** $1/(e^x+1)$	2 **15** $e^{1/x}+(1/e^x)$	2 **16** $e^{\sqrt{x}}+\sqrt{e^x}$
2 **7** $e^{\sqrt{x+1}}$	**8** xe^{-x}	**17** $\dfrac{e^x-e^{-x}}{e^x+e^{-x}}$	**18** $e^{x \ln x}$
9 x^2e^{-2x}	1 **10** $\sqrt{e^{2x}+2x}$	**19** $e^{-2x} \ln x$	**20** $\ln e^x$

Finally, we solve for y' as follows:

$$y' - (x^2 \cos y)y' = 2x \sin y$$
$$(1 - x^2 \cos y)y' = 2x \sin y$$
$$y' = \frac{2x \sin y}{1 - x^2 \cos y},$$

provided $1 - x^2 \cos y \neq 0$.

When we study *Taylor series* in Chapter 11, it will be necessary to find many higher derivatives of functions. The next example illustrates that these derivatives are easy to find for the sine function.

EXAMPLE 11 Find the first eight derivatives of $f(x) = \sin x$.

SOLUTION Applying (8.11) yields

$$f'(x) = D_x \sin x = \cos x$$
$$f''(x) = D_x \cos x = -\sin x$$
$$f'''(x) = D_x(-\sin x) = -D_x(\sin x) = -\cos x$$
$$f^{(4)}(x) = D_x(-\cos x) = -D_x(\cos x) = -(-\sin x) = \sin x.$$

Since $f^{(4)}(x) = \sin x$, it follows that if we continue differentiating, the same pattern repeats; that is,

$$f^{(5)}(x) = \cos x, \quad f^{(6)}(x) = -\sin x,$$
$$f^{(7)}(x) = -\cos x, \quad f^{(8)}(x) = \sin x.$$

EXERCISES 8.2

Exer. 1–54: Find the derivative.

1 $f(x) = 4\cos x$

2 $H(z) = 7\tan z$

3 $G(v) = 5v \csc v$

4 $f(x) = 3x \sin x$

5 $f(\theta) = \dfrac{\sin\theta}{\theta}$

6 $g(\alpha) = \dfrac{1 - \cos\alpha}{\alpha}$

7 $g(t) = t^3 \sin t$

8 $T(r) = r^2 \sec r$

9 $h(z) = \dfrac{1 - \cos z}{1 + \cos z}$

10 $R(w) = \dfrac{\cos w}{1 - \sin w}$

11 $p(x) = \sin x \cot x$

12 $g(t) = \csc t \sin t$

13 $f(x) = \dfrac{\tan x}{1 + x^2}$

14 $h(\theta) = \dfrac{1 + \sec\theta}{1 - \sec\theta}$

15 $k(v) = \dfrac{\csc v}{\sec v}$

16 $q(t) = \sin t \sec t$

2 17 $k(x) = \sin(x^2 + 2)$

1&2 18 $f(t) = \cos(4 - 3t)$

1&2 19 $H(\theta) = \cos^5 3\theta$

1&2 20 $g(x) = \sin^4(x^3)$

21 $g(z) = \sec(2z + 1)^2$

22 $k(z) = \csc(z^2 + 4)$

23 $H(s) = \cot(s^3 - 2s)$

24 $f(x) = \tan(2x^2 + 3)$

1&2 25 $f(x) = \cos(3x^2) + \cos^2 3x$

26 $g(w) = \tan^3 6w$

27 $F(\phi) = \csc^2 2\phi$

28 $M(x) = \sec(1/x^2)$

29 $K(z) = z^2 \cot 5z$

30 $G(s) = s \csc(s^2)$

31 $h(\theta) = \tan^2\theta \sec^3\theta$

32 $H(u) = u^2 \sec^3 4u$

33 $N(x) = (\sin 5x - \cos 5x)^5$

34 $p(v) = \sin 4v \csc 4v$

35 $h(w) = \dfrac{\cos 4w}{1 - \sin 4w}$

36 $f(x) = \dfrac{\sec 2x}{1 + \tan 2x}$

37 $f(x) = \tan^3 2x - \sec^3 2x$

38 $h(\phi) = (\tan 2\phi - \sec 2\phi)^3$

39 $f(x) = \sin\sqrt{x} + \sqrt{\sin x}$